CLUSTER ALGEBRAS EXERCISES
$\mathrm{Gr}(3,6)$, $\mathrm{Gr}(3,7)$, $\mathrm{Gr}(3,8)$

Background

The coordinate ring of Grassmannians have a nice cluster algebra structure. A seed which can be built for any $\mathrm{Gr}(k,n)$ is the *rectangle seed*. The following appears in [1, Chapter 6].

Let $Q_{k,n}$ be a quiver with vertices labeled by rectangles contained in a $k \times (n-k)$ rectangle, plus the empty rectangle. We freeze the vertices of $Q_{k,n}$ corresponding to the empty rectangle and to rectangles with one of its sides of maximal size: i.e. of shape $k \times \star$ or $\star \times (n-k)$. In 2 the frozen vertices are the ones in blue. For a rectangle $i \times j$ we draw an arrow to the rectangles of sizes $(i+1)\times j$, $i\times(j+1)$ and $(i-1)\times(j-1)$ as long as these rectangles still fit in a $k\times(n-k)$ rectangle, are non-empty and do not join two frozen vertices.

The fact that the quiver $Q_{k,n}$ gives rise to a cluster algebra structure in $\mathbb{C}[\widehat{\mathrm{Gr}}(k,n)]$ is not a trivial result; the proof can be found in [1, Chapter 6] and [2]. For today we will just explain how to go from a rectangle to a Plücker coordinate. Given a rectangle R we left-top justify it on top of the $k \times (n-k)$ rectangle and consider the n-steps path from the left-bottom corner to the right-top corner traced by where R cuts the bigger rectangle. Labeling all steps of the path from left to right we have that the Plücker coordinate corresponding to R is given by the labels in the vertical steps of the path.

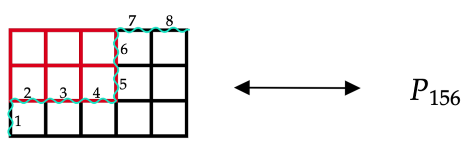


Figure 1. Example: from rectangle to Plücker coordinate

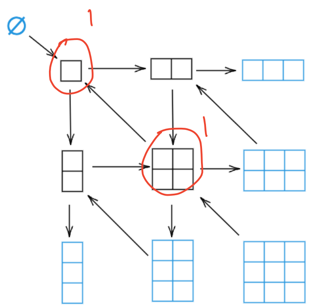
In today's exercises we will explore some special cases of what are known as cluster algebras of *finite type*. These correspond to cluster algebras such that for at least one of its seed the corresponding quiver (removing the frozen vertices) forms a Dynkin diagram.

Exercises

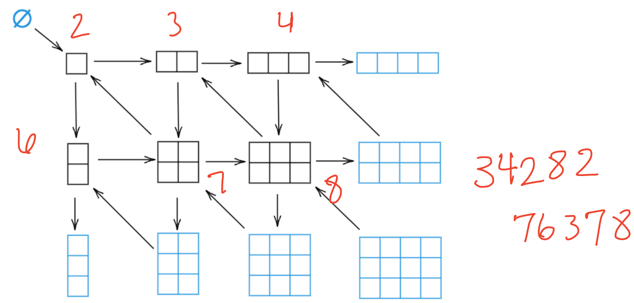
Exercise 1: The three Grassmannians $\mathrm{Gr}(3,n)$ with $n = 6,7,8$ correspond to cluster algebras of finite type. Find among the six Dynkin diagrams in Figure 3 the Dynkin diagram to which each Grassmannian corresponds.

For this exercise we have given you in Figure 2 the quiver corresponding to the rectangle seed for each case. You can also use the online resources in:
`public.websites.umich.edu/~fomin/cluster.html`
or `zngzag42.github.io/ClusterLibrary/index.html`.

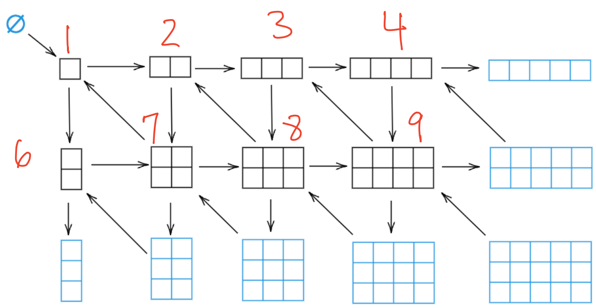
Date: March 3rd 2025.



(A) Gr(3, 6)



(B) Gr(3, 7)



E6
3,4,2,8,2,7,6,3,7,8

E8
2,3,1,8,2,7,6,3,7,8,2,3,2,1,4,7,3,4,2,3,4,7,9,2,4,6,1,7,4,8,7,6,7,4,9

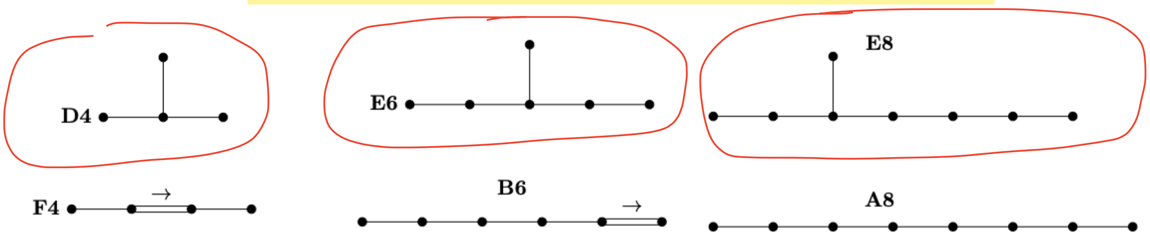


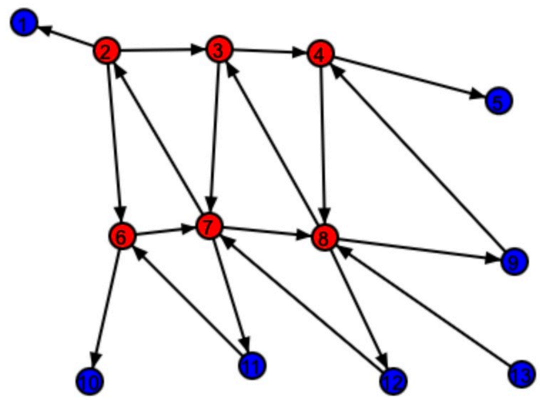
FIGURE 3. Some Dynkin diagrams

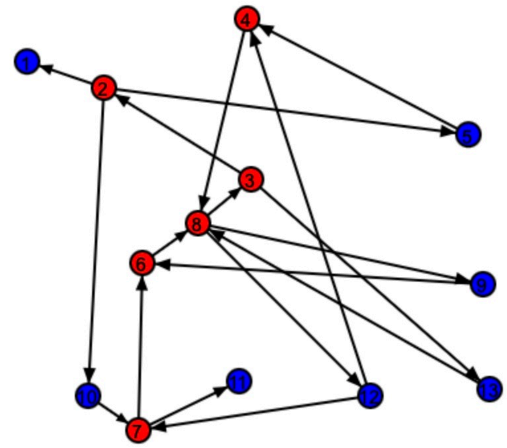
Exercise 2: Find among the cluster variables of $\mathrm{Gr}(3, n)$ with $n = 6, 7, 8$ the variables which are not Plücker coordinates. Can you give a geometric interpretation of these cluster variables? (*Hint:* represent a point in $\mathrm{Gr}(3, n)$ as a matrix and look at its columns as points in $\mathbb{P}^2$.)

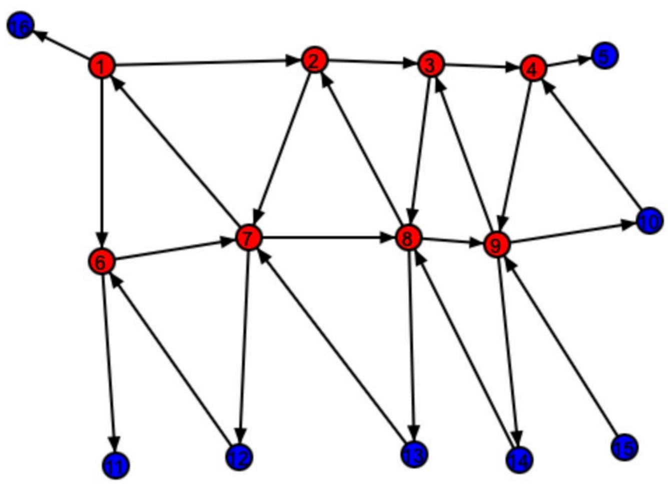
Exercise 3 (An option if you don't care about Grassmannians): Consider triangulations of an annulus with 1 marked point in the interior boundary and 2 mark points in the exterior boundary. Compute the exchange graph, given by going from one triangulation to the other, and the set of quiver isomorphism classes.

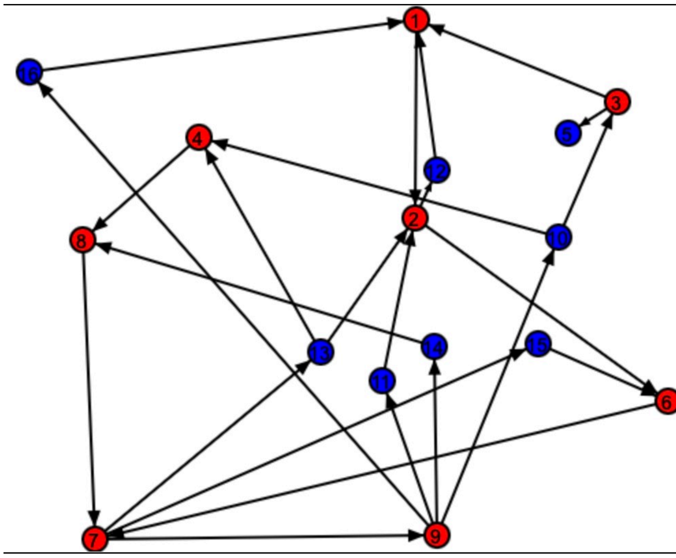
REFERENCES

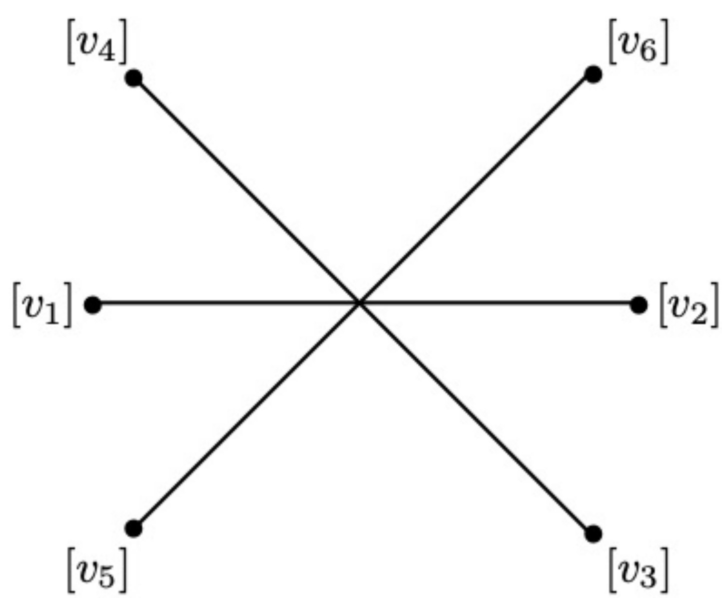
[1] Fomin S., Williams L., and Zelevinsky A. (2021) *Introduction to Cluster Algebras*, arxiv.org/abs/1608.05735.
[2] Scott J. (2003) *Grassmannians and Cluster Algebras*, arxiv.org/abs/math/0311148.











$$Y^{123456}(v_1, v_2, v_3, v_4, v_5, v_6) \; = \; X^{123456}(v_6, v_1, v_2, v_3, v_4, v_5)$$

Theorem 7. *The cluster algebra $\mathbb{C}\Big[\mathbb{G}(3,7)\Big]$ possesses forty two cluster variables. Twenty eight of these are the Plücker coordinates Δ^{ijk} - where $\{i,j,k\} \subset [1\dots 7]$ is an internal 3-subset - and the remaining fourteen are the quadratic regular functions $X^{[1\dots 7]-i}$ and $Y^{[1\dots 7]-i}$ defined above for $i \in [1\dots 7]$.*

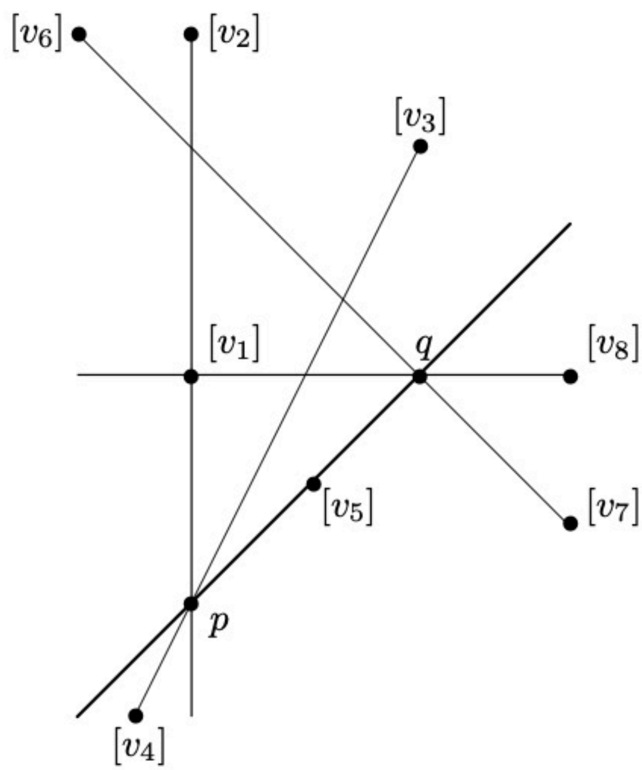


FIGURE 18. Vanishing Locus of A

Theorem 8. *The cluster algebra $\mathbb{C}\Big[\mathbb{G}(3,8)\Big]$ possesses 128 cluster variables. Of these forty eight are Plücker coordinates Δ^{ijk} - where $\{i,j,k\} \subset [1\dots n]$ is an internal 3-subset. Fifty six cluster variables are the quadratic regular functions $X^{[1\dots 8]-\{ij\}}$ and $Y^{[1\dots 8]-\{ij\}}$ - for $1 \le i < j \le 8$ - inherited from $\mathbb{G}(3,6)$. The remaining twenty four are dihedral translates of two cubic regular functions, denoted as A and B. The zero locus of A consists of configuration of eight projective points $[v_1],\dots,[v_8]$ for which the points p, q and $[v_5]$ are colinear (see illustration below).*