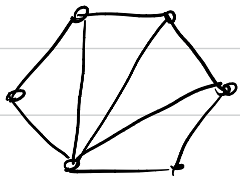
General Cluster Algebras

Recall: Cluster algebra for triangulated surface

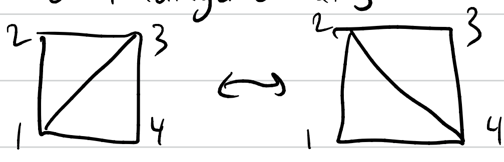
Triangulated Surface



$\rightsquigarrow$ seed, variable for each arc

triangulation

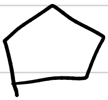
Two triangulations are related by a "flip"



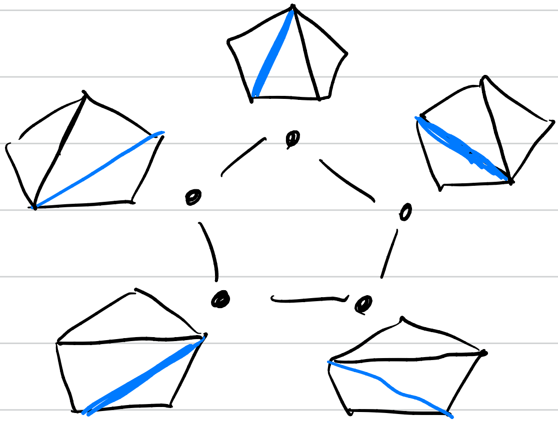
$$13 \cdot 24 = 12 \cdot 34 + 14 \cdot 23$$

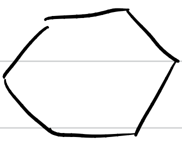
- Consolidate this structure in exchange complex
  - 0-cell for each triangulation
  - 1-cell for each flip

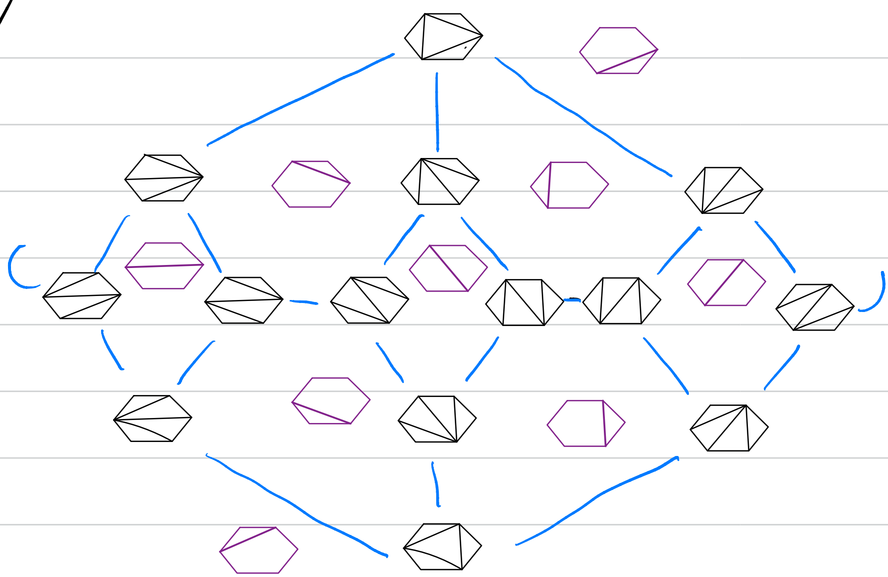
ex) 

ex) 

Exchange complex



ex) 



Want Generalization that gives structure to $Gr(k,n)$, higher teichmüller spaces, etc.

① Replace triangulation with a quiver

<u>Defn</u>: A quiver Q is directed graph with no 2 cycles or self loops. (Picture of skew symmetric matrix)

<u>ex)</u> $1 \to 2 \to 3$ $\qquad \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ $\qquad$ <u>ex)</u> $1 \Rightarrow 2$ $\qquad \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

- <u>Exchange Matrix</u> B_{ij} is adjacency matrix of Q

i.e. B_{ij} = (signed) # of arrows from i to j

② Associate variable to each node of Q

<u>Defn</u>: A <u>seed</u> is the pair of a quiver Q and list of variables $(a_1, \dots, a_n)$ for each node.

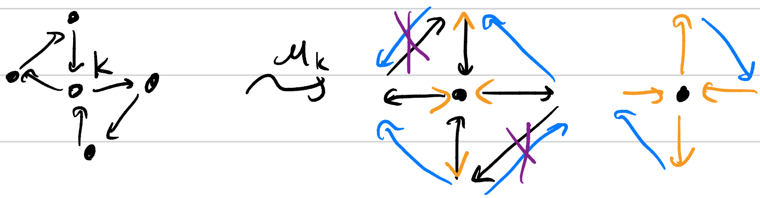
Mutation

- We want an involution defined at each node of Q, that "mirrors" the flip

Defn: <u>Quiver mutation</u>: Given a node k of Q we obtain a new quiver $\mu_k(Q)$ via the following

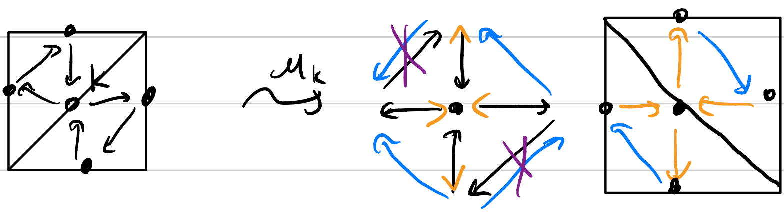
1. For each path $i \to k \to j$ in Q add $i \to j$
2. Remove any 2-cycles
3. Reverse all arrows incident to k

ex)



ex)

- Can obtain a quiver from a triangulation by taking a node for each arc and oriented cycle for each triangle



- <u>Remark</u>: Not all arcs of triangulation can be flipped. In particular boundary arcs always present.

→ Modify definition of a seed to include a partition of the nodes of Q into frozen and unfrozen/mutable
  - Graphically draw frozen nodes as $\square$, mutable $\bullet$

- <u>Exchange relation</u>: $a_k \cdot \mu(a_k) = \prod_{i \to k} a_i + \prod_{k \to j} a_j$

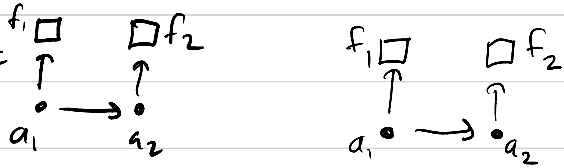
- matches triangulation exchange rule.

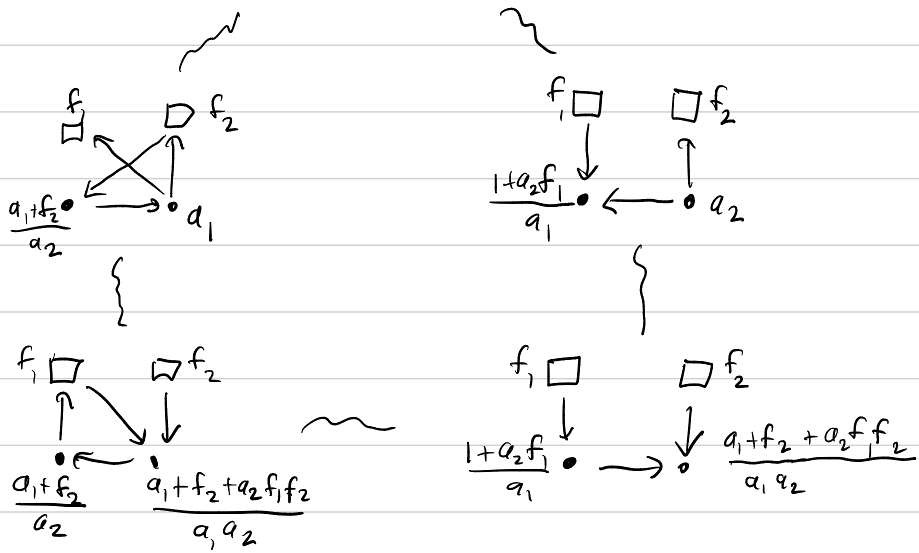
Cluster Algebra

Let $(Q, (a_1,\dots,a_n, a_{n+1},\dots,a_{n+m}))$ be the <u>initial seed</u>

- Abstractly $a_1,\dots, a_{n+m} \in k(a_1,\dots,a_{n+m})$

The cluster algebra $\mathcal{A}_Q$ <u>grown from</u> $(Q,\vec{a})$ is the subalgebra generated by all cluster variables obtained by arbitrary sequences of mutations

ex] $Q=$ 



$$\mathcal{A}_Q = \mathbb{Z}\left[a_1, a_2, \frac{1+a_2f_1}{a_1}, \frac{a_1+f_2}{a_2}, \frac{a_1+f_2+a_2f_1f_2}{a_1a_2}\right] \subseteq \mathbb{Z}(a_1,a_2,f_1,f_2)$$

Thm: If Q and Q' are mutation equivalent, then $\mathcal{A}_Q \cong \mathcal{A}_{Q'}$. Moreover the exchange complex is isomorphic as well.

Thm: Laurent Phenomena: Every cluster variable can be expressed as a Laurent Polynomial in the initial seed.
→ True for any seed so cluster variables are "universally Laurent", $\mathcal{A}_Q \subseteq \bigcap_{Q' \sim Q} k[\vec{a}_{Q'}^{\pm}]$

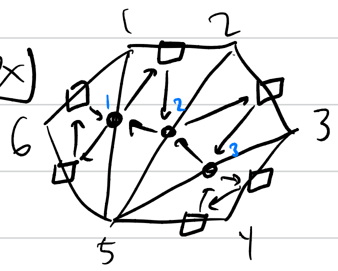
For general cluster algebra there are other universally Laurent functions

→ Moreover frozen variables never appear in denominator
Numerator has positive integer coefficients

X Coordinates

Defn: An <u>X-coordinate</u> at mutable node k is $\prod_{k \to j} a_j \Big/ \prod_{i \to k} a_i$

ex)



$$X_1 = \frac{P_{12} P_{56}}{P_{16} P_{25}}$$

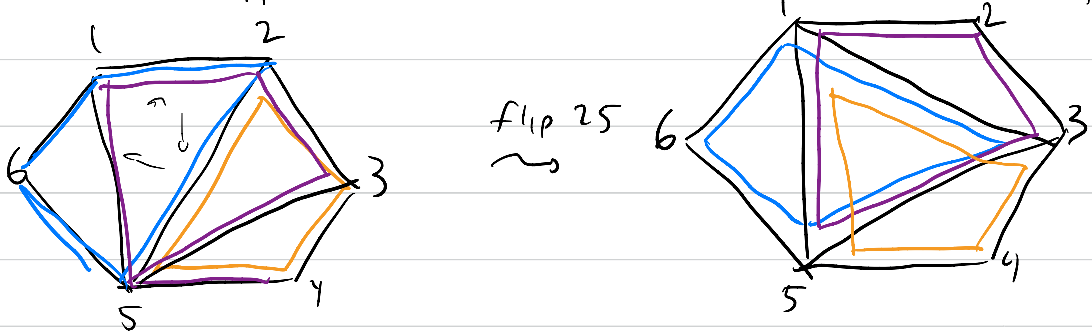
Remark: This ratio is invariant under

1. changing horocycle / torus action
2. Projective transformation

Related to usual projective cross ratio (only slightly different convention)

→ name X-coordinate is for "cross X"

What happens to X coordinates when we flip



$X_1 = \frac{12 \cdot 56}{25 \cdot 16}$ $X_2 = \frac{23 \cdot 15}{35 \cdot 12}$ $X_3 = \frac{34 \cdot 25}{23 \cdot 45}$

$X_1' = \frac{13 \cdot 56}{35 \cdot 16}$ $X_2' = \frac{35 \cdot 12}{23 \cdot 15}$ $X_3' = \frac{34 \cdot 15}{13 \cdot 45}$

Solve

$X_1' = \frac{13 \cdot 25}{12 \cdot 35} X_1$ $X_3' = \frac{15 \cdot 23}{13 \cdot 25} X_3$

= exchanged arcs = neighbors / factors of X_2

Key Trick: $1 + X_2 = \frac{12 \cdot 35 + 23 \cdot 15}{35 \cdot 12} = \frac{13 \cdot 25}{12 \cdot 35}$, $1 + X_2^{-1} = \frac{13 \cdot 25}{15 \cdot 23}$

$X_1' = (1 + X_2) X_1$ $X_3' = (1 + X_2^{-1})^{-1} X_3$

Can use this to define an X-mutation rule without referencing A-coords

$$X_i \xmapsto{\mu_k} \begin{cases} X_i^{-1} & i=k \\ X_i(1+X_k)^{w} & k \xrightarrow{w} i \\ X_i(1+X_k^{-1})^{-w} & i \xrightarrow{w} k \end{cases}$$

• <u>Fact</u>: Get same exchange complex if use A or X mutation

<u>Remark</u>: there are "more" X coordinates than normal (A) coordinates

The map $\mathbb{C}[\mathcal{X}\text{-variety}] \longrightarrow \mathbb{C}[\mathcal{A}\text{-variety}]$

$$x_i \longmapsto \prod a_j^{b_{ij}}$$

is not injective or surjective in general

<u>ex)</u>

$$\begin{tikzcd} \overset{a_1}{\bullet} \arrow[r] & \overset{a_2}{\bullet} & \overset{a_3}{\bullet} \arrow[l] \end{tikzcd}$$

$x_1 = a_2 \quad x_3 = a_2$ not injective

$x_2 = \frac{1}{a_1 a_3}$

– Can't get a_1 or a_3 by themselves.

But can add frozen variables to make injective