Last Time

Cluster Algebra Alphabet

- A coordinates
- B exchange Matrix (adjacency matrix of quiver)

C-vector, connections to "coefficients"/frozen nodes

e - standard basis

f - polynomial

g - vector

Information from a framed seed that describes cluster variables in other equivalent algebras

X - coordinate "cross ratio"

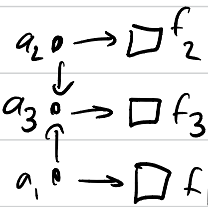
mutation affects neighbors

y - frozen part of X coordinate - should be "viewed tropically"

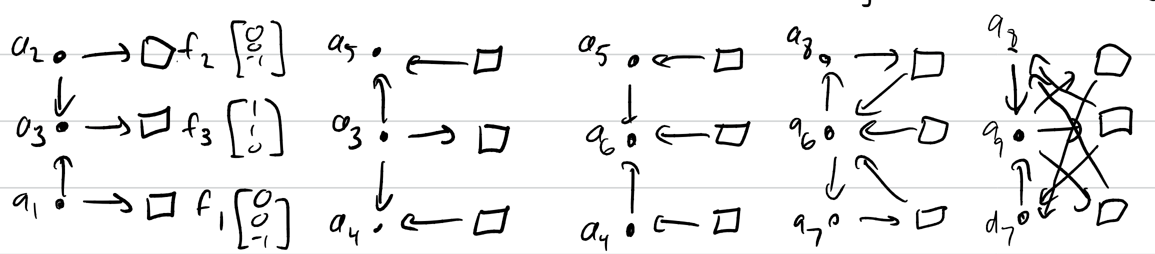
G-vector Exchange: Tropical Mutation

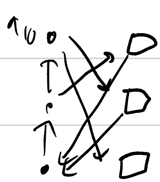
$$g_k' = -g_k + \min\left(\sum_{i \to k} g_i, \sum_{k \to j} g_j\right)$$

F-polynomial Exchange: Usual exchange relation (A-coord) with initial values 1

ex)  $= Q$ $\quad B = \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$

name	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
F-poly	1	1	1	$1+f_1$	$1+f_2$	$1+(1+f_1)(1+f_2)f_3$	$1+(1+f_2)f_3$	$1+(1+f_1)f_3$	$1+f_3$
G-vec	$\begin{bmatrix}1\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\1\end{bmatrix}$	$\begin{bmatrix}-1\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\-1\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\-1\end{bmatrix}$	$\begin{bmatrix}1\\0\\-1\end{bmatrix}$	$\begin{bmatrix}0\\1\\-1\end{bmatrix}$	$\begin{bmatrix}1\\1\\-1\end{bmatrix}$





$$g_{vec}(a_{10}) = -\begin{bmatrix}0\\1\\-1\end{bmatrix} + \min\left(\begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix} = g_{vec}(a_1)$$

Thm: Let $(Q,\vec{a})$ be initial seed of cluster algebra, Let $\hat{Q}$ be framed quiver with same mutable part as Q. Consider a sequence of mutations in cluster algebra resulting in new cluster variable a'. The expression of a' as Laurent polynomial in $(Q,\vec{a})$ is

$$a' = a_1^{g_1}\cdots a_n^{g_n}\,\frac{F(x_1,\dots,x_n)}{F_{\mathbb{P}}(y_1,\dots,y_n)}$$

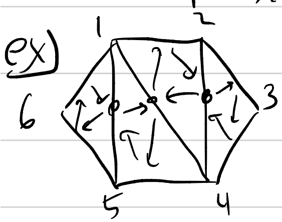
where a_i = initial a variables in $(Q,\vec{a})$

x_i = initial X-variables as ratio of a vars $= \prod a_j^{b_{ij}}$

y_i = initial coefficients (frozen X-variables) $= \prod_{j=1}^{m} f_j^{b_{i(n+j)}}$

and $F, \vec{g}$ are the F-polynomial and g-vector computed by same sequence of mutations in framed seed

ex)



$a_1 = 15 \quad a_2 = 24 \quad a_3 = 14$

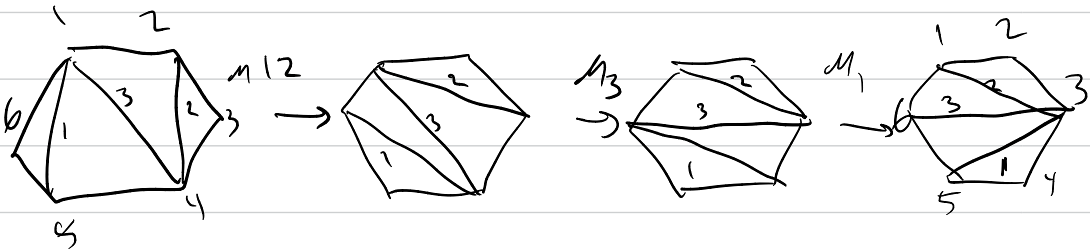
$x_1 = \frac{14\cdot 56}{45\cdot 16} \quad x_2 = \frac{14\cdot 23}{12\cdot 34} \quad x_3 = \frac{12\cdot 45}{15\cdot 24}$

$y_1 = \frac{56}{45\cdot 16} \quad y_2 = \frac{23}{12\cdot 34} \quad y_3 = 12\cdot 45$

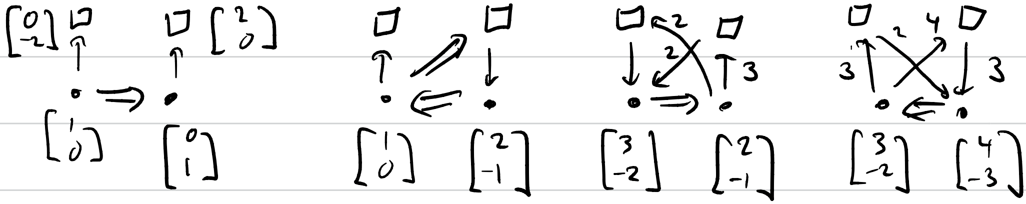
$$a_7 = a_1^1 a_2^0 a_3^{-1}\,\frac{(1+(1+x_2)x_3)}{(1\oplus(1\oplus y_2)y_3)} = \frac{15}{14}\,\frac{\left(1+\left(1+\frac{14\cdot 23}{12\cdot 34}\right)\left(\frac{12\cdot 45}{15\cdot 24}\right)\right)}{\left(\frac{1}{34}\right)} =$$

$$= \frac{\cancel{15}\cdot\cancel{34}}{14}\,\frac{\cancel{12}\cdot 34\cdot 15\cdot 24 + \cancel{12}\cdot 45\cdot 12\cdot 34 + 14\cdot 23\cdot\cancel{12}\cdot 45}{\cancel{12}\cdot\cancel{34}\cdot\cancel{15}\cdot 24} = \frac{34\cdot 15\cdot 24 + 12\cdot 34\cdot 45 + 14\cdot 23\cdot 45}{14\cdot 24}$$

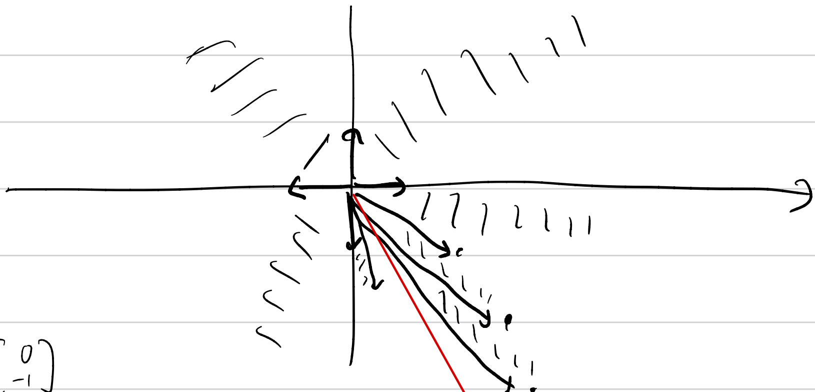
$$= \frac{34\cdot\cancel{14}\cdot 25 + \cancel{14}\cdot 23\cdot 45}{\cancel{14}\cdot 24} = \frac{24\cdot 35}{24} = 35$$


μ12
μ3
μ1

ex) G-vector only



Get G-vector Fan



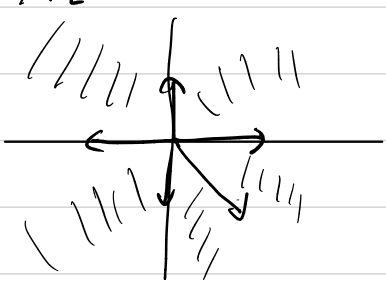
$\begin{bmatrix}-1\\0\end{bmatrix} \Rightarrow \begin{bmatrix}0\\-1\end{bmatrix}$

$\begin{bmatrix}1\\-2\end{bmatrix}$

missing/accumulation ray

$\begin{bmatrix}n\\-n\end{bmatrix}$

Contrast with A_2



Covers all of $\mathbb{R}^2$

This difference is sympotem of bigger property

$\bullet \to \bullet$ is finite

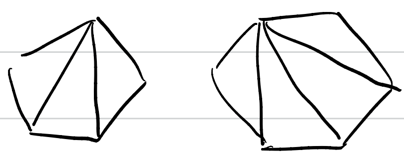
$\bullet \rightrightarrows \bullet$ is infinite.

__Defn__: A cluster algebra is __finite type__ if it contains finitely many cluster variables

$\iff$ it contains finitely many seeds

Otherwise we say it is infinite type.

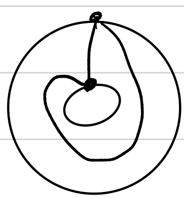
__ex)__



... any polygon

$\bullet \to \bullet$ $\qquad$ $\bullet \to \bullet \to \bullet$

__Not Finite__: $\bullet \rightrightarrows \bullet$



How Do You Detect Finite Type

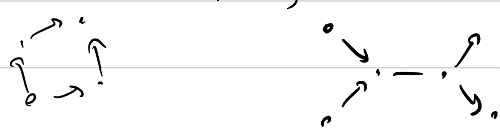
<u>Claim</u>: If Q contains an arrow of weight higher than 1, the associated cluster algebra is infinite

<u>Proof</u>: Look at the cluster subalgebra generated by the two nodes $\bullet \xrightarrow{w} \bullet$. This has infinite variables

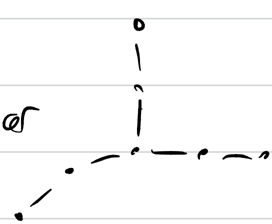
- Cluster subalgebra is cluster algebra given by freezing nodes in any seed of bigger algebra

<u>Cor</u>: If Q is mutation equivalent to a node with a high weight edge then the cluster algebra is infinite

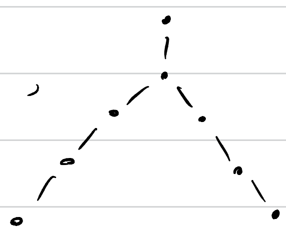
<u>Cor</u>: If Q contains unoriented cycle, double fork

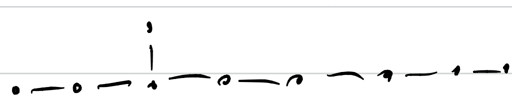


or

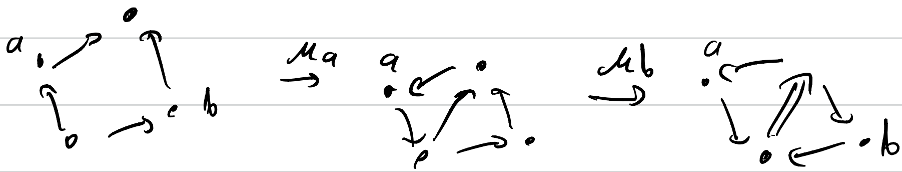


,

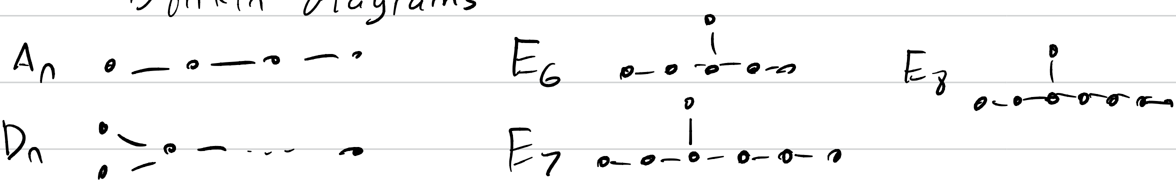




Proof: Can find mutation sequence to a double edge

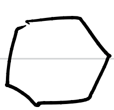
ex 

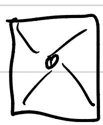
We recognize these graphs from Lie theory
They are the <u>affine Dynkin Diagrams</u>
The forbidden graphs that define the finite (simply laced) Dynkin diagrams



Claim: If Q is (mutation equivalent) any orientation of a Dynkin diagram then the cluster algebra $\mathcal{A}_Q$ is finite

Proof Sketch 1

$A_n =$  $n+3$ gon

$D_n =$  Punctured n-gon

Check E_6, E_7, E_8 by computer