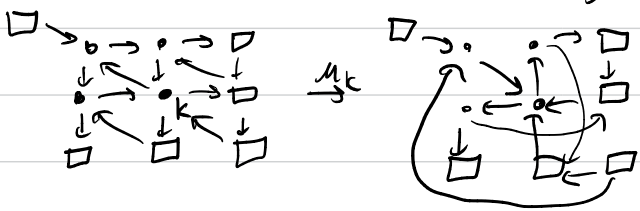
Review

Seed: quiver + set of variables
- mutable/unfrozen vs frozen
- Exchange relation $\mu_k(a_k) \cdot a_k = \prod_{i \to k} a_i + \prod_{k \to j} a_j$

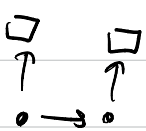
Quiver mutation

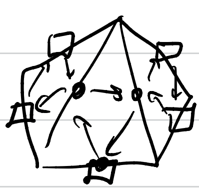


Cluster Algebra generated by all possible mutations.

Defn: Exchange graph: vertex for each seed,
edge for each mutation

Last time saw example of 2 different seeds with same mutable part that produce same exchange graph.



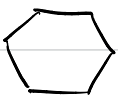


Also saw "X- exchange Relation"

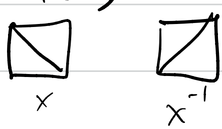
$$X_i \xmapsto{\mu_k} \begin{cases} X_i^{-1} & i=k \\ X_i(1+X_k)^{w} & k \xrightarrow{w} i \\ X_i(1+X_k^{-1})^{-w} & i \xrightarrow{w} k \end{cases}$$

• Fact: Get same exchange complex if use A or X mutation starting from the same quiver

Remark: There are "more" X coordinates than normal (A) coordinates

ex| In  there are $\binom{6}{2} - 6 = 9$ mutable A coords (one for each internal edge)

There are $2 \cdot \binom{6}{4} = 15 \cdot 2$ X-coordinates (one for each "square")

with diagonal 

There is a map $\mathbb{C}[\mathcal{X}\text{-variety}] \longrightarrow \mathbb{C}[\mathcal{A}\text{-variety}]$

$$x_i \longmapsto \prod a_j^{b_{ij}}$$

is not injective or surjective in general

ex)

$$\begin{tikzcd} \overset{a_1}{\bullet} \arrow[r] & \overset{a_2}{\circ} & \overset{a_3}{\circ} \arrow[l] \end{tikzcd}$$

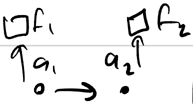
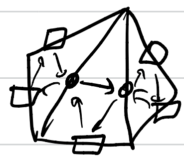
$x_1 = a_2 \qquad x_3 = a_2 \qquad$ not injective

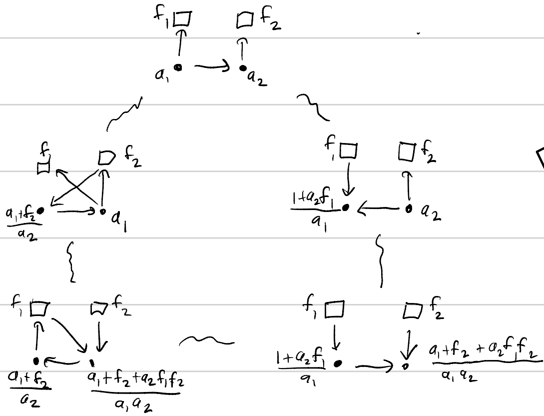
$x_2 = \frac{1}{a_1 a_3}$

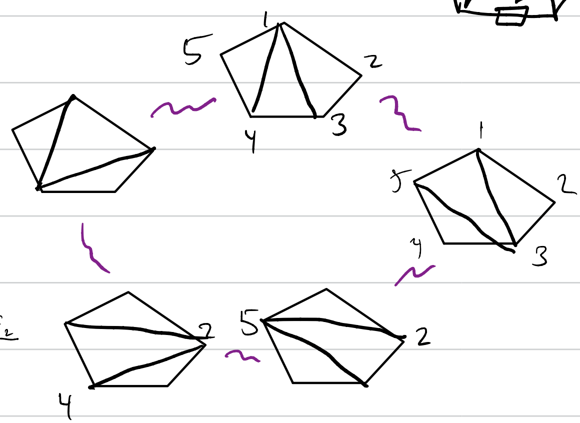
- can't get a_1 or a_3 by themselves.

But can add frozen variables to make injective

Changing Frozen Variables

Recall  has same exchange graph as 



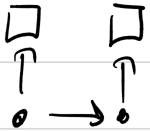
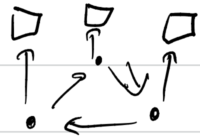


Remark: The quivers have the same mutable part

Thm: The exchange complex of a quiver is independent of frozen variables

Proof: Key idea: Can chose a set of frozen variables which "specializes" to any other choice

Defn: A framing of a quiver Q is quiver $\hat{Q}$ with one frozen node for each mutable node attached 'out'

ex)  

Defn: The c-vector (coefficient vector) of a mutable node k is vector whose i^{th} entry is the # of arrows from k to frozen variable $i+n$

Remark: If Q has frozen/unfrozen nodes the exchange matrix has the form

$$\hat{B} = \left[\begin{array}{c|c} B & \begin{matrix} - c_1 - \\ \vdots \\ - c_n - \end{matrix} \\ \hline \begin{matrix} | & & | \\ -c_1 & \cdots & -c_n \\ | & & | \end{matrix} & \star \end{array}\right]$$

where B is exchange matrix of mutable part

$\tilde{B} = [\, B \mid c \,]$ is called <u>extended exchange matrix</u>

- entries in $\star$ don't affect mutation so are ignored (Sometimes chose half edges to explain gluing behavior)

Thm: If $\hat{Q}$ is a framed quiver then the c-vectors are sign coherent, each vector is nonpositive or nonnegative

Sign coherence lets us write a mutation rule for c-vectors as follows

$$\mu_k c_i = \begin{cases} -c_k & \text{if } i=k \\ c_i + w\, c_k & \text{if } c_k \geq 0 \quad i \xrightarrow{w} k \\ c_i + (-w\, c_k) & \text{if } c_k \leq 0 \quad k \xrightarrow{w} i \end{cases}$$

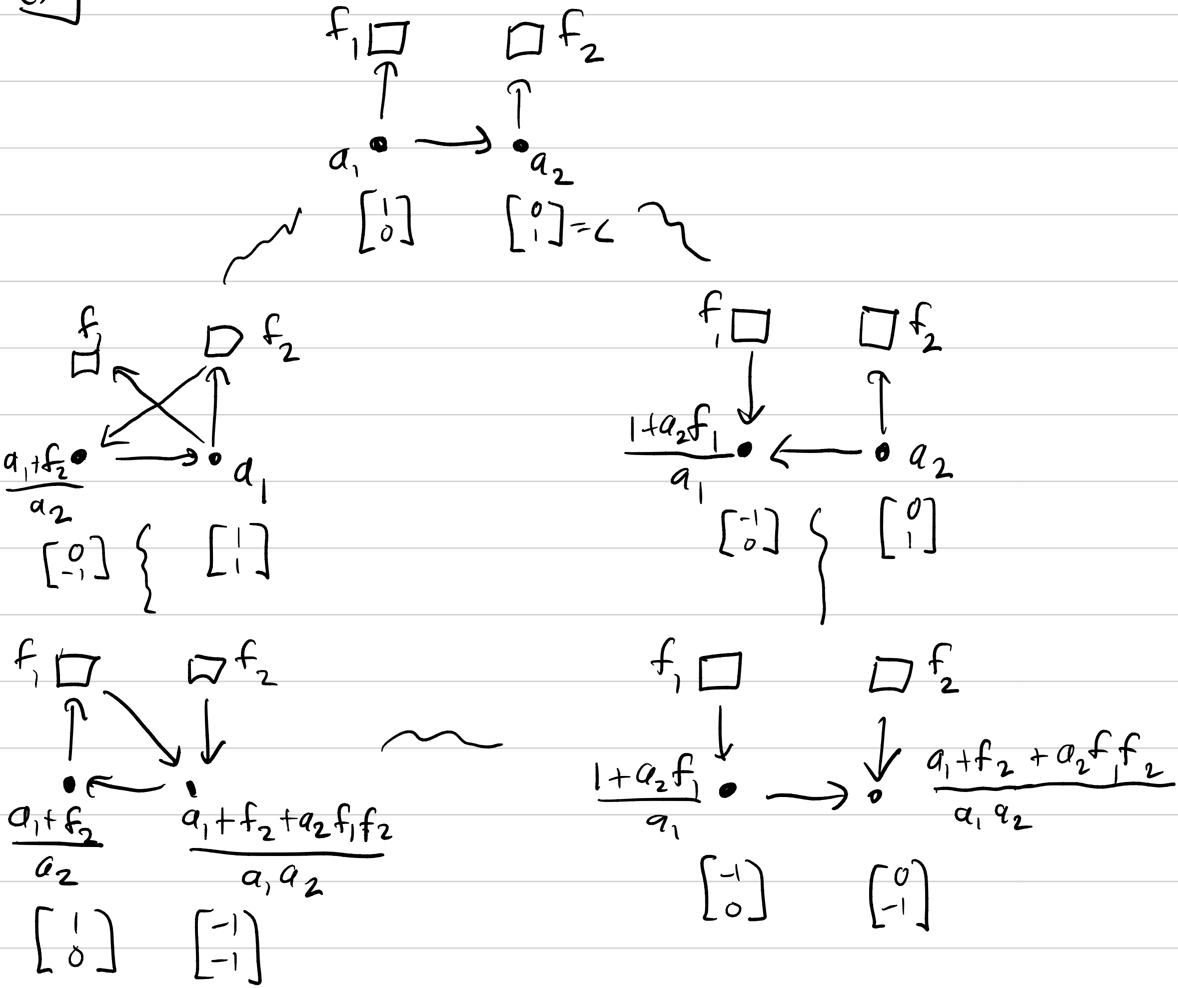
$$= \begin{cases} -c_k & \text{if } i=k \\ c_i + (1 \oplus w\, c_k) & \text{if } i \xrightarrow{w} k \\ c_i - (1 \oplus -w\, c_k) & \text{if } k \xrightarrow{w} i \end{cases}$$

This is "tropical X mutation"

Defn: The tropical semifield $\mathrm{Trop}(q_1, \dots, q_n)$ has set

Laurent Monomials $\prod q_i^{v_i}$	Vectors $[v_1, \dots, v_n]$
$\otimes$ usual mult	$\otimes$ vector addition
$\oplus$ $\prod q_i^{v_i} \oplus \prod q_i^{w_i} = \prod q_i^{\min(v_i, w_i)}$	$\oplus$ Pointwise minimum

ex



f_1
f_2
a_1
a_2
$\begin{bmatrix}1\\0\end{bmatrix}$
$\begin{bmatrix}0\\1\end{bmatrix} = c$
$\frac{a_1+f_2}{a_2}$
$\begin{bmatrix}0\\-1\end{bmatrix}$
$\begin{bmatrix}1\\1\end{bmatrix}$
$\frac{1+a_2f_1}{a_1}$
$\begin{bmatrix}-1\\0\end{bmatrix}$
$\begin{bmatrix}0\\1\end{bmatrix}$
$\frac{a_1+f_2+a_2f_1f_2}{a_1a_2}$
$\begin{bmatrix}1\\0\end{bmatrix}$
$\begin{bmatrix}-1\\-1\end{bmatrix}$
$\begin{bmatrix}-1\\0\end{bmatrix}$
$\begin{bmatrix}0\\-1\end{bmatrix}$

Defn: The <u>F-polynomial</u> is the polynomial obtained from framed cluster variables by setting $x_i = 1$.

ex)

a_1	1
a_2	1
$1 + a_2 f_1 / a_1$	$1 + f_1$
$a_1 + f_2 + a_2 f_1 f_2 / a_1 a_2$	$1 + f_2 + f_1 f_2$
$a_1 + f_2 / a_2$	$1 + f_2$

Thm: Each F-polynomial has constant term 1

Proof: Equivalent to sign-coherence [GHKK]

Gross-Hacking-Keel-Kontsevich

strong Laurent Phenomena implies there are no frozen variables in denominator $\rightsquigarrow$ these are actual polynomials with positive coefficients

<u>Thm</u>: Each cluster variable can be expressed as

$$\hat{a}(a_1,\dots,a_n,f_1,\dots,f_m) = \frac{a_1^{g_1}\cdots a_n^{g_n}\, F(x_1,\dots,x_n)}{F_{\mathbb{P}}(y_1,\dots,y_n)}$$

where $y_i = \prod_{j=1}^{m} f_j^{b_{ij}}$ $\quad x_i$ = x-coord at $i = \prod a_j^{b_{ij}}$

the vector $g_1 \cdots g_n$ is called <u>g-vector</u> of $\hat{a}$

$\mathbb{P}$ = Tropical Semifield on n-generators (Laurent monomials)

<u>ex)</u> $x_1 = a_2 f_1 \quad x_2 = f_2/a_1 \quad y_1 = f_1 \quad y_2 = f_2$

$$\begin{tikzcd} \boxed{f_1} & \boxed{f_2} \\ a_1 \arrow[u] \arrow[r] & a_2 \arrow[u] \end{tikzcd}$$

<u>Remark</u>: The g-vector is the (multi) degree of $\hat{a}$ with respect to the weighting $\deg(a_i) = e_i$ $\deg(f_i) = B^{\circ}_{\cdot,i}$

= i^{th} col of exchange

<u>ex)</u>

$$\begin{tikzcd} \boxed{f_1} & \boxed{f_2} \\ a_1 \arrow[u] \arrow[r] & a_2 \arrow[u] \end{tikzcd}$$

$\deg(a_1) = \begin{bmatrix}1\\0\end{bmatrix}$ $\quad \deg(f_1) = \begin{bmatrix}0\\-1\end{bmatrix}$

$\begin{bmatrix}0&1\\-1&0\end{bmatrix} = B^{\circ}$ $\quad \deg(a_2) = \begin{bmatrix}0\\1\end{bmatrix}$ $\quad \deg(f_2) = \begin{bmatrix}1\\0\end{bmatrix}$

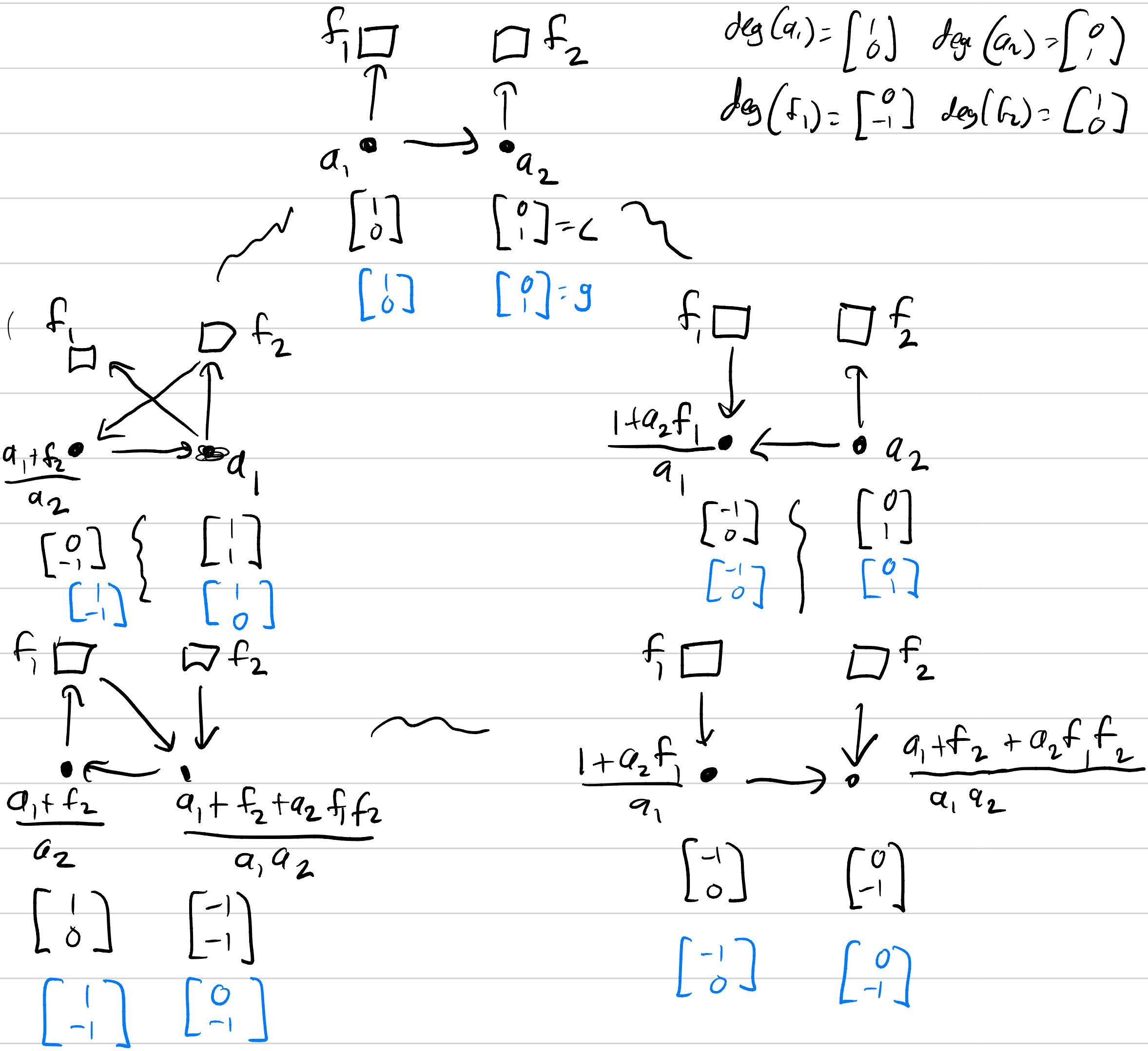
<u>ex)</u>

Clust	F	$F(x_1,x_2)$	$F_{\mathbb{P}}(y_1,y_2)$	g
a_1	1	1	1	$[1,0]$
a_2	1	1	1	$[0,1]$
$1+a_2f_1/a_1$	$1+f_1$	$1+a_2f_1$	1	$[-1,0]$
$a_1+f_2+a_2f_1f_2/a_1a_2$	$1+f_2+f_1f_2$	$1+\frac{f_2}{a_1}+\frac{a_2f_1f_2}{a_1}$	1	$[0,-1]$
a_1+f_2/a_2	$1+f_2$	$1+f_2/a_1$	1	$[1,-1]$

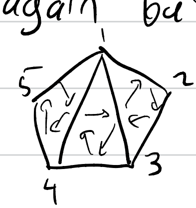
$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\deg(a_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \deg(a_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\deg(f_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \deg(f_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



ex 2 | Now we compute A_2 again but with the quiver from the triangulation



. So $a_1 = 14$
$a_2 = 13$

To use formula's we need to compute

$$X_1 = \frac{13 \cdot 45}{34 \cdot 15} \qquad X_2 = \frac{12 \cdot 34}{23 \cdot 14} \qquad y_1 = \frac{45}{15 \cdot 34} \qquad y_2 = \frac{12 \cdot 34}{23}$$

Then $a_3 = a_1^{-1} a_2^{0} (1 + X_1)/(1 \oplus y_1)$

$$(14)^{-1}(13)^{0} \frac{\left(1 + \frac{13 \cdot 45}{34 \cdot 15}\right)}{1 \oplus \frac{45}{15 \cdot 34}} = \frac{\cancel{15 \cdot 34}}{14}\left(\frac{34 \cdot 15 + 13 \cdot 45}{\cancel{34 \cdot 15}}\right) = \frac{1}{\cancel{14}}\left(\cancel{14} \cdot 35\right) = 35$$

$$a_4 = (14)^{0}(13)^{-1} \frac{\left(1 + \frac{12 \cdot 34}{23 \cdot 14} + \frac{13 \cdot 45 \cdot 12}{15 \cdot 23 \cdot 14}\right)}{1 \oplus \frac{12 \cdot 34}{23} \oplus \frac{12 \cdot \cancel{34}}{23} \cdot \frac{45}{15 \cdot \cancel{34}}}$$

$$= \frac{1}{13} \cancel{23} \cdot \cancel{15} \left(\frac{15 \, 13 \cdot 24}{\cancel{15}\, \cancel{23} \cdot 14} + \frac{13 \cdot 45 \cdot 12}{\cancel{15} \cdot \cancel{23} \cdot 14}\right) = \frac{1}{\cancel{13}}\left(\frac{\cancel{13}(\cancel{14} \cdot 25)}{\cancel{14}}\right) = 25$$

$$a_5 = (14)^{1}(13)^{-1} \frac{\left(1 + \frac{12 \cdot 34}{23 \cdot 14}\right)}{\left(1 \oplus \frac{12 \cdot 34}{23}\right)} = \frac{\cancel{14}\ \cancel{23}}{\cancel{13}}\left(\frac{\cancel{13} \cdot 24}{\cancel{23} \cdot \cancel{14}}\right) = 24$$

In light of formula $\hat{a}(\ ) = \vec{a}^{\vec{g}} \cdot F(\vec{x})/F_{\mathbb{P}}(\vec{y})$ we think of g vector as "tropicalization" of cluster coordinates

Given a quiver Q' obtained by mutation from Q
Can we obtain the g-vector recursively?

Mutate at k: $g_k' = -g_k + \min\left(\sum_{i\to k} g_i, \sum_{k\to j} g_j\right)$

This is "tropical mutation"
- Remark: $\sum_{i\to k} g_i = \sum_{k\to j} g_j$ by construction of degree (frozen degree chosen to balance)

Thm (Nakanishi–Zelevinsky: On tropical Dualities in cluster algebras) Thm 1.2: For framed quiver Q, and a seed Q' obtained by mutation

$$C_{Q'}^{-1} = G_{Q'}^{T}$$

Finishing the Alphabet

Defn: The d-vector (Denominater vector) is vector of powers of mutable variables in the denominater of Laurent expansion.

- Note: as F-poly has constant term 1 implies numerator has constant term 1 ⇒ tropical d-vector rule

Summary

A - coordinates (Cluster variables)
B - exchange matrix = adjacency matrix of quiver
C - vectors "coefficient vectors"
D - vector "Denominater vector
e_i - standard basis

$$\left.\begin{array}{l}\text{F - polynomial} \\ \text{G - vector}\end{array}\right\}\ \text{combine to give cluster variable}$$

⋮

Q = quiver

⋮

X - coordinate