Cluster Algebras From Triangulated Surfaces

Goal: Give motivating examples of varieties/rings with a cluster structure.

<u>Ex 1</u>: Grassmannian of k-planes in n dimensional space

$$Gr(k,n) = \{ V \subseteq \mathbb{R}^n (\mathbb{C}^n) \mid \dim(V) = k \}$$
$$= GL_k(\mathbb{R}) \backslash Mat(k \times n, \mathbb{R})$$

• <u>Plücker Embedding</u>: $Gr(k,n) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$

$M \mapsto (P_I(M))$ where $P_I = \det(M_{i_1}, \ldots, M_{i_k})$

Check: Multiplication by change of basis B changes

$(P_I) \longmapsto \det(B) \cdot P_I$

Today focus is on $Gr(2,n) \cong$ Configurations of n pts in $\mathbb{P}^1$

• <u>Plücker Relations</u>: Choose $i<j<k<l$

$$P_{ik} P_{jl} = P_{ij} P_{kl} + P_{il} P_{jk}$$

- For $Gr(2,4)$ one relation $P_{13} P_{24} = P_{12} P_{34} + P_{14} P_{23}$

- On $Gr(2,4)$ Plücker relations come from characterization of image

Defn: $x \in \Lambda^k V$ is totally decomposable $x = v_1 \wedge \dots \wedge v_k$
- this means x is in image of Plücker embedding

Fact: x totally decomposable then $x \wedge x = 0$

If expand in a basis $x = \sum a_I e_{i_1} \wedge \dots \wedge e_{i_k}$
- when x totally decomposable $a_I = P_I(v_1, \dots, v_k)$

In $Gr(2,4)$ $x \wedge x = (P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23})\, e_1 \wedge e_2 \wedge e_3 \wedge e_4$

So to be 0 need $P_{12}P_{34} + P_{14}P_{23} = P_{13}P_{24}$ ✓

- More generally get relations by considering $\varphi_x : V \to \Lambda^{k+1} V$
  $\dim(\ker(\varphi_x)) = k$ iff x totally decomposable $\quad v \mapsto v \wedge x$

- <u>Affine Cone</u>: $\widehat{Gr(k,n)} = SL_k(\mathbb{R}) \backslash Mat(k \times n, \mathbb{R})$
  - embeds in affine space via Plücker embedding

Q: ① What sets of Plücker coords can we freely specify?

② Can we make all coords positive?

③ Are there natural projectively invariant functions?

Teichmüller Space of Bordered Surface

- **Bordered Surface:** Σ is orientable surface with boundary marked points and punctures
  - Want "total curvature" negative so there is a triangulation of Σ

$$\tilde{\chi}(\Sigma) = 2-2g-p-b-\tfrac{1}{2}n$$

g = genus b = # boundary components p = # punctures n = # marked pts

Lemma: # triangles $= -2\tilde{\chi}(\Sigma)$

$$\tilde{\chi}(\triangle) = 2-2\cdot 0-0-1-\tfrac{3}{2} = -\tfrac{1}{2}$$

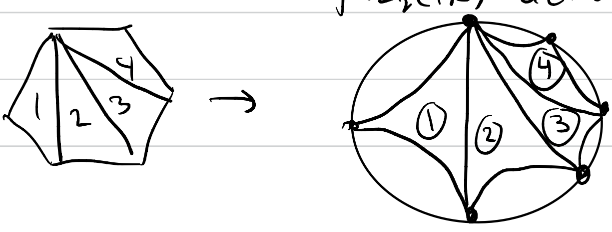
Today: Focus on $g=0$, $p=0$, $b=1$, i.e. polygons with n sides

$$\tilde{\chi}(\Sigma) = 2-0-1-0-\tfrac{1}{2}n \Rightarrow n \geq 3$$

Defn: Teichmüller space $\mathcal{T}(\Sigma) = \{$finite area hyperbolic metrics on Σ with marked pts cusps, geodessic boundary$\}$

- take metrics up to diffeomorphism which are homotopic to the identity relative to fixing marked points

Defn: Developing map: Embed Σ in $\mathbb{H}^2$. Can work triangle by triangle since unique ideal triangle up to $PSL_2(\mathbb{R})$ action

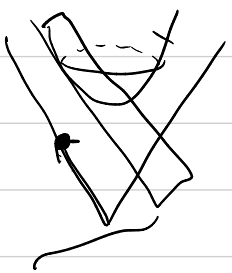


For a disk this shows hyperbolic structure is equivalent to choice of n cyclically ordered ideal points (in $\partial \mathbb{H}^2 = \mathbb{P}^1$) up to action of $PSL_2(\mathbb{R})$

☆ This connects $Gr(2,n) \sim \mathcal{T}(\text{⬠})$

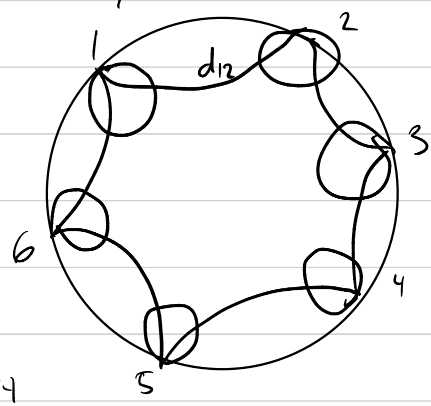
→ What does affine cone $\widehat{Gr(2,n)}$ correspond to?

<u>Decorated Teichmüller Space</u>: Teichmüller space + choice of horocycle at each marked point.



$P = \{x \mid q(x,v) = -1\}$

quadratic form   vector at infinity



$\rightarrow \lambda_{12} = e^{d_{12}/2}$

Lambda lengths

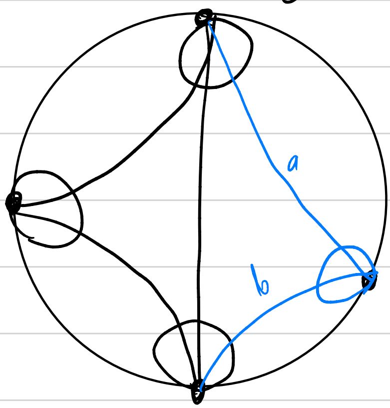
Normalize s.t. $|v_i \wedge v_j| = \lambda_{ij}$. So $P_{ij} = \lambda_{ij}$

- horocycle at v in light cone is $\langle w, v \rangle = -1$.

Plücker relation $\leftrightarrow$ Hyperbolic Ptolemy Relation

Question Answers

① What sets of coordinates can we freely choose



triangulation → defines decorated configuration

Relations require crossing arcs!

② Can we make all coordinates positive?

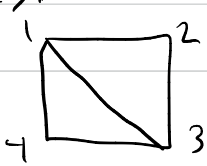
Yes. In hyperbolic model coords are defined positive

⇒ True for $Gr(2,n)$, we will see later for $Gr(k,n)$

The totally positive grassmannian $Gr^{>}(k,n)$ is set where all Plücker coordinates are positive

③ Are there natural projectively invariant functions?

Yes. To a square have cross ratio



$$X = \frac{P_{12}\,P_{34}}{P_{14}\,P_{23}} = \frac{(b_2-b_1)(b_4-b_3)}{(b_4-b_1)(b_3-b_2)}$$

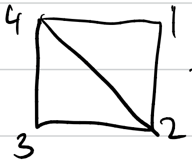
- Inspired by cross ratio Fock/Goncharov call these X coords.
  They call usual cluster variables A-coordinates.

• well defined on projective Grassmannian

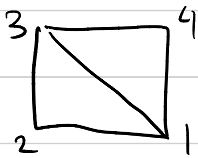
$$\frac{d\,P_{12}\; d\,P_{34}}{d\,P_{14}\cdot d\,P_{24}} = \frac{P_{12}\,P_{34}}{P_{14}\,P_{24}} \quad \Big\}\ \text{True for any degree 0 ratio.}$$

$\star$ This isn't exactly cross ratio usually defined in hyperbolic geometry. $\left(\frac{(b_3-b_1)(b_4-b_2)}{(b_3-b_2)(b_4-b_1)}\right)$

This ratio behaves nicely under cyclic rotate

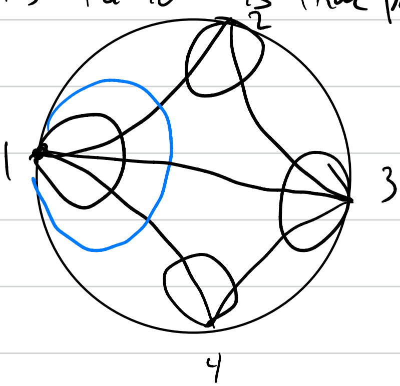


$= X^{-1}$



$= X$

• This ratio is independent of horocycle/torus action

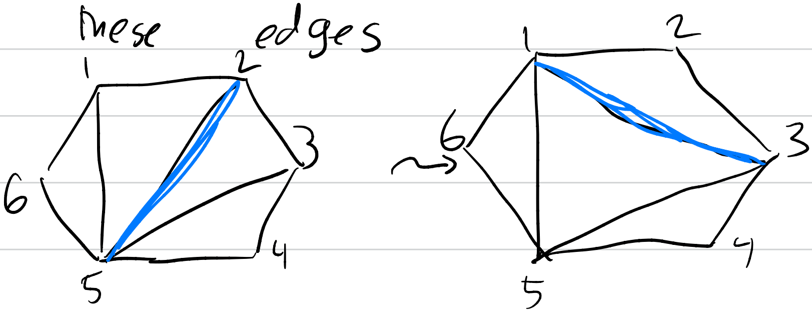


- change to blue changes P_{12} and P_{14} by same factor
- equivalent to scaling v_1 by t

$$\frac{\cancel{t}\,P_{12}\cdot P_{34}}{\cancel{t}\,P_{14}\cdot P_{24}}$$

Structure of Algebra

- Each triangulation consists of independent functions
- Have relation whenever two edges cross
  This relates the triangulations given by swapping these edges



$$P_{25} P_{13} = P_{12} P_{35} + P_{23} P_{15}$$

<u>Final Fact</u>: Any 2 triangulations are related by a series of flips inside a square

- Hatcher – On triangulations of Surfaces 1991

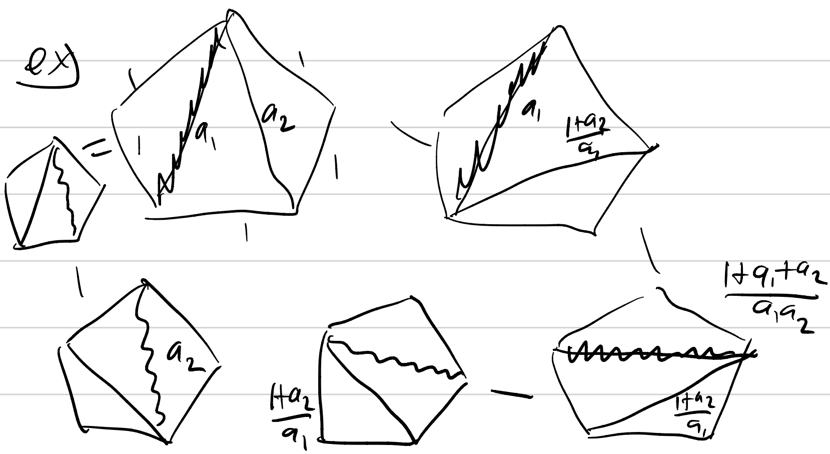
Properties of Algebra

- Exchange Relation only use addition and multiplication
  - Positivity: If all variables are positive in one seed all variables are positive.
  - "Tropical": Can understand exchange in any semiring
    ex) $(\mathbb{Z}, +, \max) \rightsquigarrow$ leading term degree

- ex)

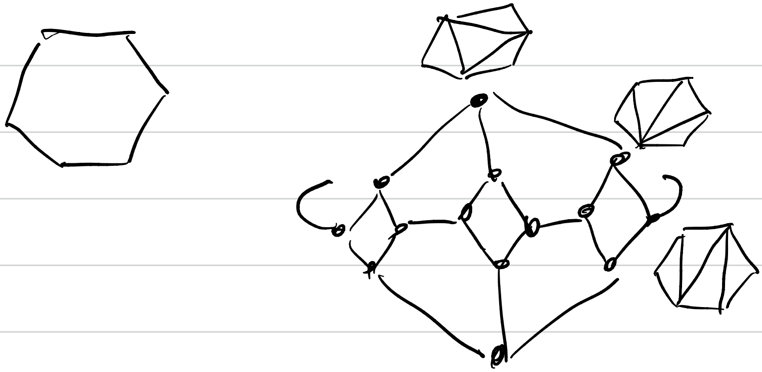
Two seeds exactly

ex)



$\rightarrow$ Observe Laurent Phenomena.

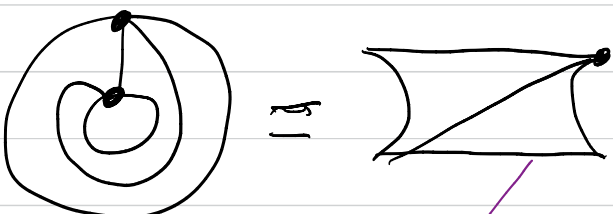
ex

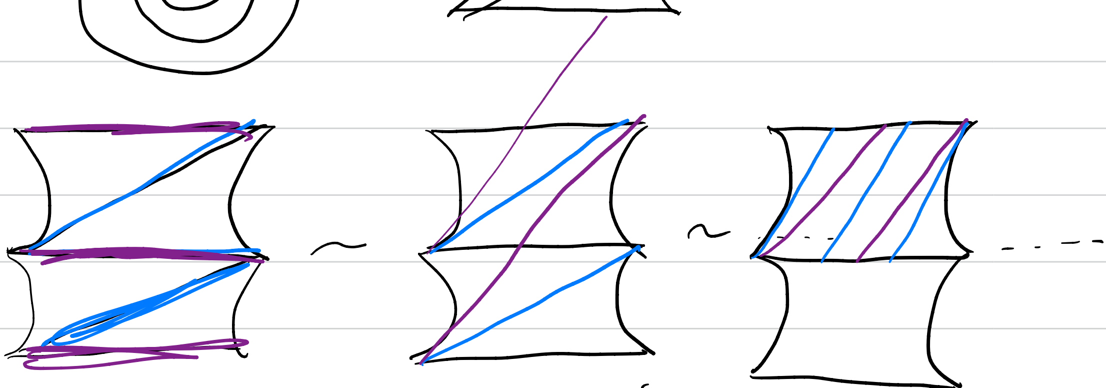


Cluster Exchange Complex

- Node for each seed (0-cell)
- Edge for each mutation (1-cell)
- Face for each rank 2 subalgebra (2 cell)

ex





Infinetly many triangulations

Questions For Future

- When are there finitely many seeds?
- What are symmetries of cluster complex?
- What other algebras/varieties have cluster structure?
- How do we generalize triangulation to cover $Gr(k,n)$?